

# Seesaw Mechanism in Three Flavors

Haijun Pan<sup>1\*</sup>, and G. Cheng<sup>2,3†</sup>

<sup>1</sup>*Lab of Quantum Communication and Quantum Computation, and Center of Nonlinear Science,  
University of Science and Technology of China, Hefei, Anhui, 230026, P.R.China*

<sup>2</sup>*CCAST(World Laboratory), P.O.Box 8730, Beijing 10080, P.R.China*

<sup>3</sup>*Department of Astronomy and Applied Physics,  
University of Science and Technology of China, Hefei, Anhui, 230026, P.R.China*

## Abstract

We advance a method used to analyse the neutrino properties (masses and mixing) in the seesaw mechanism. Assuming quark-lepton symmetry and hierarchical light neutrino masses, we establish rather simple relations between the light and the heavy neutrino parameters in the favored regions of the solar and the atmospheric neutrino experiments. A empirical condition satisfied by the right-handed mixing angles is obtained.

PACS numbers: 14.60.Pq, 12.15.Ff

Typeset using REVTeX

---

\*Email: phj@mail.ustc.edu.cn

†Email: gcheng@ustc.edu.cn

## I. INTRODUCTION

Whether neutrinos have nonzero masses or not? How large would the mixing angles be? Are they like that in the quark sector? Those are among the pressing questions in particle physics. The solar [1] and atmospheric [2] neutrino data suggest that neutrinos do have masses and the recent results from Super-Kamiokande (SK) [2] imply a nearly maximal mixing of  $\nu_\mu$  and  $\nu_\tau$ . In another hand, the fact that neutrinoless double- $\beta$  decay and other lepton number nonconserving processes are not observed experimentally reflects the smallness of the neutrino masses [3]. The seesaw mechanism [4] has a natural explanation for the small neutrino masses and may enhance lepton mixing up to maximal [5–7].

According to the seesaw mechanism, at  $M \gg m_D$ , the Majorana mass matrix  $m^{\text{eff}}$  of the left-handed (LH) neutrino components is given as [5]

$$m^{\text{eff}} = m_D M^{-1} m_D^T. \quad (1)$$

Here  $M$  is the Majorana mass matrix of the right-handed (RH) neutrino components and  $m_D$  is the neutrino Dirac mass matrix which could be equal to the mass matrix of the up quarks:  $m_D = m^{up}$  according to some kind of quark-lepton symmetry [5,6,8]. In the basis where  $M^{-1}$  is diagonal,  $M^{-1} = M_i^{-1} \delta_{ij} \equiv R_i^2 \delta_{ij}$  ( $i, j = 1, 2, 3$ )  $m_D$  can be written as [8]

$$m_D = U_0 m_D^{\text{diag}} V_0. \quad (2)$$

Here  $U_0$  and  $V_0$  are LH and RH rotations respectively and  $m_D^{\text{diag}} = \text{diag} \{m_1, m_2, m_3\}$ .

In this paper, we study a problem what we can know about the masses and mixing of the right handed neutrinos from the low energy neutrino data. The paper is organized as follows. In Sec. II a parameterization is introduced and the seesaw mechanism is expressed in two formula: one of them involves only the neutrino masses and the other involves only some nondimensional parameters such as mass ratios and mixing angles. Then the RH neutrino masses and mixing angles are derived. In Sec. III we get rather simple relations between the masses and mixing angles entering the seesaw formula in the favored regions of the solar

and atmospheric experiments. The numerical results they infer are given whereafter. We summarize and discuss our main results in Sec. IV.

## II. GENERAL FRAMEWORK

### A. Parameterization

Since the CP-violating effects in neutrino oscillations should be small [9], we shall therefore ignore it and so  $U_0$  and  $V_0$  are real orthogonal matrices. For simplicity, We also set  $U_0 \sim I$ . That is, the left-handed rotations that diagonalize the charged lepton  $m_1$  and neutrino Dirac mass matrices  $m_D$  are the same or nearly the same and so the large lepton mixing results from the seesaw transformation [5]. Under these assumptions, it is convenient to write

$$m_D^{\text{diag}} V_0 M^{-1} V_0^T m_D^{\text{diag}} = U_0^T U \left( N^{\text{diag}} \right)^2 U^T U_0 \approx U \left( N^{\text{diag}} \right)^2 U^T, \quad (3)$$

or by inverting it,

$$\left( m_D^{\text{diag}} \right)^{-1} U \left( N^{\text{diag}} \right)^2 U^T \left( m_D^{\text{diag}} \right)^{-1} = V_0 M^{-1} V_0^T \quad (4)$$

where  $U$  is LH rotation induced by  $M^{-\frac{1}{2}}$  and  $N^{\text{diag}} = \text{diag} \{n_1, n_2, n_3\}$  with  $n_i^2 = m_i^{\text{eff}}$  ( $i = 1, 2, 3$ ), the eigenvalues of  $m^{\text{eff}}$ .

Analogy with that in the two flavors case [10], we introduce the following mass parameterization,

$$\xi_3 = \frac{1}{2} \ln \frac{m_2}{m_1}, \quad \xi_8 = \frac{1}{6} \ln \frac{m_3^2}{m_1 m_2}; \quad (5a)$$

$$\eta_3 = \frac{1}{2} \ln \frac{R_1}{R_2}, \quad \eta_8 = \frac{1}{6} \ln \frac{R_1 R_2}{R_3^2}; \quad (5b)$$

$$\kappa_3 = \frac{1}{2} \ln \frac{n_2}{n_1}, \quad \kappa_8 = \frac{1}{6} \ln \frac{n_3^2}{n_1 n_2} \quad (5c)$$

and the mixing matrices are parameterized as usual,

$$U = \exp(i\theta_{23}\lambda_7) \exp(i\theta_{13}\lambda_5) \exp(i\theta_{12}\lambda_2), \quad (6a)$$

$$V_0 = \exp(i\beta_{23}\lambda_7) \exp(i\beta_{13}\lambda_5) \exp(i\beta_{12}\lambda_2). \quad (6b)$$

Here,  $\lambda_2, \lambda_5, \lambda_7$  are Gell-mann matrix. One can see that  $\eta_3$  and  $\eta_8$  describe the hierarchy of the RH neutrino masses and are always nonnegative. Especially,  $\eta_3 = 0$  implies  $M_1 = M_2$  while  $\eta_3 = 3\eta_8$  implies  $M_2 = M_3$ . Using the diagonal Gell-mann matrix  $\lambda_3$  and  $\lambda_8$ , the mass matrices involving now can be rewritten as

$$m_D^{\text{diag}} = (m_1 m_2 m_3)^{\frac{1}{3}} e^{-\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8}, \quad (7a)$$

$$M^{-1} = (R_1^2 R_2^2 R_3^2)^{\frac{1}{3}} e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8}, \quad (7b)$$

$$(N^{\text{diag}})^2 = (n_1^2 n_2^2 n_3^2)^{\frac{1}{3}} e^{-2\kappa_3 \lambda_3 - 2\sqrt{3} \kappa_8 \lambda_8}. \quad (7c)$$

This parameterization shows clearly that the relevant variables in the diagonalization of  $M^{-1}$  are  $\theta_{12}, \theta_{13}, \theta_{23}, \kappa_3, \kappa_8, \xi_3$  and  $\xi_8$ . Of these, it is usually assumed that  $\xi_3$  and  $\xi_8$  can be identified with the corresponding quantities of the up sector of quarks as stated before and  $\theta_{12}, \theta_{13}, \theta_{23}, \kappa_3, \kappa_8$  can be obtained, at least approximately, from the low energy neutrino data. Now let us denote

$$\begin{aligned} \bar{X}(\kappa, \xi, \theta) &= (R_1^2 R_2^2 R_3^2)^{\frac{1}{3}} V_0 e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} V_0^T \\ &= (m_1 m_2 m_3)^{-\frac{2}{3}} (n_1^2 n_2^2 n_3^2)^{\frac{1}{3}} X(\kappa, \xi, \theta). \end{aligned} \quad (8)$$

Here

$$X(\kappa, \xi, \theta) = e^{\xi_3 \lambda_3 + \sqrt{3} \xi_8 \lambda_8} U e^{-2\kappa_3 \lambda_3 - 2\sqrt{3} \kappa_8 \lambda_8} U^T e^{\xi_3 \lambda_3 + \sqrt{3} \xi_8 \lambda_8} \quad (9)$$

and  $\kappa, \xi$  and  $\theta$  refer to  $\kappa_3, \kappa_8; \xi_3, \xi_8$  and  $\theta_{12}, \theta_{13}, \theta_{23}$  respectively. Eq. (8) is equivalent with the following two equations:

$$R_1^2 R_2^2 R_3^2 = (m_1 m_2 m_3)^{-2} (n_1^2 n_2^2 n_3^2), \quad (10a)$$

$$X(\kappa, \xi, \theta) = V_0 e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} V_0^T. \quad (10b)$$

The first relation is just the equality of the determinations of both sides of Eq. (8). Taking the total term  $(R_1^2 R_2^2 R_3^2)^{\frac{1}{3}} = (m_1 m_2 m_3)^{-\frac{2}{3}} (n_1^2 n_2^2 n_3^2)^{\frac{1}{3}}$  out from Eq. (8) we get the second relation. For late use, we present here the expression of the inverse of  $X(\kappa, \xi, \theta)$ . It is easy to know from Eq. (9) that

$$X^{-1}(\kappa, \xi, \theta) = e^{-\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8} U e^{2\kappa_3 \lambda_3 + 2\sqrt{3} \kappa_8 \lambda_8} U^T e^{-\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8}. \quad (11)$$

So that

$$X^{-1}(\kappa, \xi, \theta) = X(-\kappa, -\xi, \theta) \equiv Y(\kappa, \xi, \theta). \quad (12)$$

and we have

$$Y(\kappa, \xi, \theta) = V_0 e^{-2\eta_3 \lambda_3 - 2\sqrt{3} \eta_8 \lambda_8} V_0^T. \quad (13)$$

We will start from Eqs. (10b,13) to derive the expressions of  $\eta_3$ ,  $\eta_8$  and  $V_0$ . Then from Eq. (10a)  $M_i$  ( $i = 1, 2, 3$ ) can be obtained. In following discussion, we shall omit the variables  $\kappa, \xi, \theta$  in  $X$  and  $Y$ .

## B. Determination of the Majorana masses

In this subsection we deduce two equations about the hierarchy,  $\eta_3$  and  $\eta_8$ , of the RH neutrino masses. Taking the trace of both sides of Eq. (10b) we obtain

$$\text{Tr} \left( V_0 e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} V_0^T \right) = \text{Tr} e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} = \text{Tr} X, \quad (14)$$

that is,

$$e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} + e^{-4\eta_8} = X_{11} + X_{22} + X_{33} \equiv A. \quad (15)$$

Similarly, taking the trace of both sides of Eq. (13) we get

$$e^{-2\eta_3 - 2\eta_8} + e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} = Y_{11} + Y_{22} + Y_{33} \equiv B. \quad (16)$$

It is sufficient for solving  $\eta_3$  and  $\eta_8$  from Eqs. (15,16) since  $X_{ii}$  and  $Y_{ii}$  ( $i = 1, 2, 3$ ) are known. Once  $\eta_3$  and  $\eta_8$  are solved, inserting  $M_1 = M_3 e^{-2\eta_3 - 6\eta_8}$ ,  $M_2 = M_3 e^{2\eta_3 - 6\eta_8}$ ,  $n_1^2 = n_3^2 e^{-2\kappa_3 - 6\kappa_8}$ , and  $n_2^2 = n_3^2 e^{2\kappa_3 - 6\kappa_8}$  in Eq. (10a), we obtain the following expressions of the RH neutrino masses,

$$M_1 = F e^{-2\eta_8 - 2\eta_3}, \quad M_2 = F e^{-2\eta_8 + 2\eta_3}, \quad M_3 = F e^{4\eta_8}. \quad (17)$$

Here  $F = \frac{m_t^2}{m_3^{\text{eff}}} e^{4\kappa_8 - 4\xi_8}$  and we have identified  $m_i$  ( $i = 1, 2, 3$ ) with the masses of up quarks.

All the above results are exact but formal. We need to decouple  $\eta_3$  and  $\eta_8$  in Eqs. (15,16).

From Eq. (15), we have

$$A = e^{2\eta_3+2\eta_8} + e^{-2\eta_3+2\eta_8} + e^{-4\eta_8} \geq 3 \left( e^{2\eta_3+2\eta_8} e^{-2\eta_3+2\eta_8} e^{-4\eta_8} \right)^{\frac{1}{3}} = 3. \quad (18)$$

The equality is satisfied when  $\eta_3 = \eta_8 = 0$ , that is, when  $M_1 = M_2 = M_3$ . At  $A \gg 3$  (then  $B \gg 3$  is also true), Eq. (15) and Eq. (16) can be approximated as follows,

$$e^{2\eta_3+2\eta_8} + e^{-2\eta_3+2\eta_8} \approx A, \quad (19a)$$

$$e^{2\eta_3-2\eta_8} + e^{4\eta_8} \approx B. \quad (19b)$$

Such a case corresponds to at most two degenerate Majorana masses. There are now two possibilities to simplify the above two equations further:

(a)  $A > B$

It is easy to know from Eqs. (19a,19b) that  $A > B$  implies  $\eta_3 > \eta_8$ . So  $e^{-2\eta_3+2\eta_8} (< 1)$  may be omitted in Eq. (19a),

$$e^{2\eta_3+2\eta_8} = e^{2\eta_3-2\eta_8} e^{4\eta_8} \approx A, \quad (20)$$

Then it is easy to see from Eqs. (19b,20) that  $e^{2\eta_3-2\eta_8}$  and  $e^{4\eta_8}$  are roots of the following quadratic equation:

$$x^2 - Bx + A = 0. \quad (21)$$

and the three eigenvalues of  $X$  are

$$e^{2\eta_3+2\eta_8} \approx A, \quad (22a)$$

$$e^{-2\eta_3+2\eta_8} \approx \frac{2}{B - \sqrt{B^2 - 4A}}, \quad (22b)$$

$$e^{-4\eta_8} \approx \frac{2}{B + \sqrt{B^2 - 4A}}. \quad (22c)$$

(b)  $A < B$

In this case, we have  $\eta_3 < \eta_8$ . Omitting the term  $e^{2\eta_3-2\eta_8} (< 1)$  in Eq. (19b), we have

$$e^{4\eta_8} = e^{2\eta_3+2\eta_8} e^{-2\eta_3+2\eta_8} \approx B. \quad (23)$$

Now  $e^{2\eta_3+2\eta_8}$  and  $e^{-2\eta_3+2\eta_8}$  are roots of the following quadratic equation:

$$x^2 - Ax + B = 0. \quad (24)$$

Thus one has

$$e^{2\eta_3+2\eta_8} \approx \frac{A + \sqrt{A^2 - 4B}}{2}, \quad (25a)$$

$$e^{-2\eta_3+2\eta_8} \approx \frac{A - \sqrt{A^2 - 4B}}{2}, \quad (25b)$$

$$e^{-4\eta_8} \approx \frac{1}{B}. \quad (25c)$$

From Eq. (22) we know that  $e^{-2\eta_3+2\eta_8} \sim e^{-4\eta_8}$  (and so  $M_2 \sim M_3$ ) when  $B^2 \sim 4A$  and from Eq. (25)  $e^{2\eta_3+2\eta_8} \sim e^{-2\eta_3+2\eta_8}$  (and so  $M_1 \sim M_2$ ) when  $A^2 \sim 4B$ . Far beyond these regions, both Eq. (22) and Eq. (25) give the same asymptotic solution:  $e^{2\eta_3+2\eta_8} \approx A$ , and  $e^{-4\eta_8} \approx \frac{1}{B}$  and  $e^{-2\eta_3+2\eta_8} = e^{-2\eta_3-2\eta_8} e^{4\eta_8} \approx \frac{B}{A}$ . The solutions are also useful for rough estimation of the Majorana masses even when two of them are degenerate, which can be seen from  $e^{2\eta_3+2\eta_8} < e^{2\eta_3+2\eta_8} + e^{-2\eta_3+2\eta_8} < 2e^{2\eta_3+2\eta_8}$  and  $e^{4\eta_8} < e^{2\eta_3-2\eta_8} + e^{4\eta_8} < 2e^{4\eta_8}$ . The maximal deviations for  $e^{2\eta_3+2\eta_8}$  and  $e^{4\eta_8}$  are both 2 times.

Usually one should have to solve a cubic characteristic equation to obtain the eigenvalues. In the seesaw model, however, one usually encounters such case where  $e^{2\eta_3+2\eta_8} \gg 1$  and  $e^{-4\eta_8} \ll 1$  simultaneously. This is a practical difficulty even in numerical calculation. More worse, the solution of a cubic equation is too ugly to see any relation between various physical quantities. By taking the trace of  $X$  and its inverse, we decompose the eigen-equation in two equations and each contains the main term of  $e^{2\eta_3+2\eta_8}$  and  $e^{4\eta_8}$  respectively. In concrete calculation, the expressions of  $A$  and  $B$  can be simplified to such a great extent that the dependence on the parameters can be exhibited explicitly. We will discuss this issue later.

### C. Determination of the RH angles

Once one have the three eigenvalues solved, then the three eigenvectors (and then the three rotation angles) of  $M^{-1}$ , can be found by the standard procedure of the linear algebra.

The eigen-equation of  $X$  is

$$(X - Q_i I) \begin{pmatrix} V_{1i} \\ V_{2i} \\ V_{3i} \end{pmatrix} = 0 \quad (i = 1, 2, 3), \quad (26)$$

where  $V_{ij} = (V_0)_{ij}$  and we use  $Q_i$  ( $i = 1, 2, 3$ ) satisfying  $Q_1 > Q_2 > Q_3$  to denote the three eigenvalues of  $X$ . The eigenvectors, solution of Eq. (26), can be expressed in:

$$V_{21} = \frac{(X_{12}X_{33} - X_{13}X_{23}) - Q_1 X_{12}}{(X_{23}^2 - X_{33}X_{22}) + (X_{33} + X_{22})Q_1 - Q_1^2} V_{11}, \quad (27a)$$

$$V_{31} = \frac{(X_{13}X_{22} - X_{12}X_{23}) - Q_1 X_{13}}{(X_{23}^2 - X_{33}X_{22}) + (X_{22} + X_{33})Q_1 - Q_1^2} V_{11} \quad (27b)$$

and etc. We also know that

$$X^{-1} = \frac{1}{\det X} \text{Adjoint} X. \quad (28)$$

Notice  $\det X = 1$ , the inverse of  $X$  is just its adjoint matrix. So

$$\begin{aligned} Y_{11} &= X_{22}X_{33} - X_{23}^2, & Y_{22} &= X_{11}X_{33} - X_{13}^2, & Y_{33} &= X_{11}X_{22} - X_{12}^2, \\ Y_{12} &= X_{13}X_{23} - X_{12}X_{33}, & Y_{13} &= X_{12}X_{23} - X_{13}X_{22}, & Y_{23} &= X_{12}X_{13} - X_{11}X_{23} \end{aligned} \quad (29)$$

and  $Y_{ij} = Y_{ji}$ . The quadratic terms in Eq. (27) are just the elements of  $Y$ . By replacing them with  $Y_{ij}$  ( $i, j = 1, 2, 3$ ), we have

$$V_{21} = \frac{Y_{12} + Q_1 X_{12}}{(Y_{11} + Q_1 X_{11}) - (Q_2^{-1} + Q_3^{-1})} V_{11}, \quad (30a)$$

$$V_{31} = \frac{Y_{13} + Q_1 X_{13}}{(Y_{11} + Q_1 X_{11}) - (Q_2^{-1} + Q_3^{-1})} V_{11} \quad (30b)$$

Here we have used  $\text{Tr} X = X_{11} + X_{22} + X_{33} = Q_1 + Q_2 + Q_3$  and  $\det X = Q_1 Q_2 Q_3 = 1$ . Thus all the non-diagonal elements of  $V_0$  can be expressed in a unified form:

$$V_{ij} = \frac{Y_{ij} + Q_j X_{ij}}{(Y_{jj} + Q_j X_{jj}) - \hat{Q}_j^{-1}} V_{jj} \quad (i, j = 1, 2, 3 \text{ and } i \neq j). \quad (31)$$

Here  $\hat{Q}_j^{-1} = \text{Tr} Y - Q_j^{-1}$ . Considering the normalization condition (or unitarity of  $V_0$ )  $V_0 V_0^T = V_0^T V_0 = I$ , all the elements can be gotten from Eq. (31). Then the deduction of the three RH angles are direct:  $\tan \beta_{23} = \frac{V_{23}}{V_{33}}$ ,  $\cos \beta_{13} \sin \beta_{12} = V_{12}$ , and  $\sin \beta_{13} = V_{13}$ .



All the relations obtained, including the masses and the angles, can be easily transformed to express the light neutrino parameters in  $M^{-1}$ ,  $m_D$  and  $V_0$ . The approach is just to make the following exchange  $\kappa \leftrightarrow -\eta$ ,  $\xi \leftrightarrow -\xi$  and  $\theta_{ij} \leftrightarrow \beta_{ij}$  ( $1 \leq i < j \leq 3$ ).

### III. NEUTRINO MASSES AND MIXINGS

The deficit of muon neutrino observed by Super-Kamiokande Collaboration and the zenith angle distributions of the data can be explained by oscillation between  $\nu_\mu$  and  $\nu_\tau$  with the best-fit parameters at [2]

$$\left(\sin^2 2\theta_{23}, \Delta m_{atm}^2\right) = \left(0.95, 5.9 \times 10^{-3} \text{eV}^2\right). \quad (32)$$

The  $\nu_e - \nu_\mu$  explanation to the solar neutrino problem requires one set of the parameters (the best fit values) in Table I. corresponding to the VO, MSW (including LMA, LOW and SMA) respectively. [11]. Here MSW and VO refer to Mikheyev-Smirnov-Wolfenstein matter-enhanced oscillations [12] and vacuum oscillations (so-called just-so oscillation) respectively. LMA (SMA) stands for a large (small) mixing angle and LOW stands for low probability (or low mass). We assume the effective neutrino masses have hierarchical pattern, that is,  $m_1^{\text{eff}} \ll m_2^{\text{eff}} \ll m_3^{\text{eff}}$ . So  $n_3^2 = m_3^{\text{eff}} \approx \sqrt{\Delta m_{atm}^2}$  and  $n_2^2 = m_2^{\text{eff}} \approx \sqrt{\Delta m_{solar}^2}$ . Little is known about the value of  $m_1^{\text{eff}}$  for which we use the parameter  $r = \frac{m_2^{\text{eff}}}{m_1^{\text{eff}}} \gg 1$  to denote. In the framework of three-flavor neutrino oscillations, the big hierarchy between  $\Delta m_{atm}^2$  and  $\Delta m_{solar}^2$  together with the no observation of  $\bar{\nu}_e \rightarrow \bar{\nu}_e$  oscillation in the CHOOZ experiment implies that the  $\nu_3$ -component in  $\nu_e$  is rather small (even negligible) and the upper limit on the value of the  $\theta_{13}$  is [13]:

$$\sin^2 \theta_{13} \equiv |U_{e3}|^2 \leq 0.015 - 0.05. \quad (33)$$

We shall therefore set  $\theta_{13} = 0$ . The Dirac masses of neutrino are taken at the scale  $\mu = 10^9 \text{GeV}$  [14]:

$$m_D^{\text{diag}}(\mu) = \text{diag} \{m_u(\mu), m_c(\mu), m_t(\mu)\} = \text{diag} \{1.47 \text{MeV}, 427 \text{MeV}, 149 \text{GeV}\}. \quad (34)$$

These are the whole values entering  $A$  and  $B$ .

## IV. ANALYSIS AND RESULT

In this section we start from Eqs. (15,16) to get the RH mass hierarchies,  $\eta_3$  and  $\eta_8$ . Then using Eq. (31), the elements (and then the mixing angles) of the RH mixing matrix would be obtained. The Majorana masses can be obtained from Eq. (17).

Although we have decoupled the Majorana masses and the RH mixing, the expressions of these parameters would be so complicated due to the complicated structure of  $X$  that it is hard to see explicitly the relations of various physical parameters. The hierarchical properties of the Dirac and the effective masses of neutrinos make it possible to drop the smaller terms in  $A$  and  $B$ . In the following, only the leading order terms of  $X_{ij}$  ( $Y_{ij}$ ) and  $A$  ( $B$ ) will be reserved respectively.

Instead of calculating the RH Majorana parameters by inserting the values of these parameters in, we give a more general analysis in two cases according to whether  $\theta_{12}$  is large (VO, LMA and LOW) or small (SMA) and derive the corresponding relations between the masses and mixing of the RH neutrino and the other neutrino parameters.

### A. Case I: large $\theta_{12}$

#### 1. mass

In this case, all the elements of  $U$  have the same order except that  $U_{e3} = 0$ . Reserving the leading order terms in  $A$  and  $B$ , we find

$$A \approx U_{e2}^2 \exp(2\xi_3 + 2\xi_8 + 2\kappa_3 - 2\kappa_8) + U_{\mu3}^2 \exp(4\kappa_8 - 2\xi_3 + 2\xi_8), \quad (35a)$$

$$B \approx U_{\tau1}^2 \exp(2\kappa_3 + 2\kappa_8 + 4\xi_8). \quad (35b)$$

It is easy to see that both  $A$  and  $B$  are far larger than 3. Noticing that, when  $\frac{\Delta m_{atm}^2}{\Delta m_{solar}^2} \leq 10^8$ , we also have  $A < B$ . Then from Eq. (25) one has

$$Q_1 = e^{2\eta_3 + 2\eta_8} \approx U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8), \quad (36a)$$

$$Q_2 = e^{-2\eta_3+2\eta_8} \approx U_{\mu 3}^2 \exp(4\kappa_8 - 2\xi_3 + 2\xi_8), \quad (36b)$$

$$Q_3 = e^{-4\eta_8} \approx \frac{1}{U_{\tau 1}^2} \exp(-2\kappa_3 - 2\kappa_8 - 4\xi_8). \quad (36c)$$

Here we have used the relation  $U_{e2}^2 U_{\mu 3}^2 = U_{\tau 1}^2$  which is satisfied when  $\theta_{13} = 0$ . We would point out that our results would be right as long as  $\theta_{13}$  is small enough. Substituting the eigenvalues in Eq. (17), we have

$$M_1 \approx \frac{1}{\sin^2 \theta_{12}} \frac{m_u^2}{m_2^{\text{eff}}}, \quad M_2 \approx \frac{1}{\sin^2 \theta_{23}} \frac{m_c^2}{m_3^{\text{eff}}}, \quad M_3 \approx \sin^2 \theta_{23} \sin^2 \theta_{12} \frac{m_t^2}{m_1^{\text{eff}}}. \quad (37)$$

The formula are the same as given in Ref. [8].  $M_1$  and  $M_2$  scale as  $1/m_2^{\text{eff}}$  and  $1/m_3^{\text{eff}}$  respectively while  $M_3$  scales as  $1/m_1^{\text{eff}}$ , which gives scales for the two lighter masses,  $M_1$  and  $M_2$ , lower and the heaviest one,  $M_3$ , higher than one would expect when no mixing occurs.

## 2. angles

Reserving the leading order terms of the numerators and denominators in Eq. (31) respectively, we obtain

$$V_{21} \approx \frac{U_{\mu 2}}{U_{e2}} e^{-2\xi_3} V_{11}, \quad V_{31} \approx \frac{U_{\tau 2}}{U_{e2}} e^{-\xi_3-3\xi_8} V_{11}, \quad (38a)$$

$$V_{12} \approx -\frac{U_{\mu 2}}{U_{e2}} e^{-2\xi_3} V_{22}, \quad V_{32} \approx -\frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3-3\xi_8} V_{22}, \quad (38b)$$

$$V_{13} \approx \frac{U_{e1}}{U_{\tau 1}} e^{-\xi_3-3\xi_8} V_{33}, \quad V_{23} \approx \frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3-3\xi_8} V_{33}. \quad (38c)$$

Exploiting the unitarity of  $V_0$ , it is appropriate to set  $V_{ii} \approx 1$ . Then the three RH angles are

$$\beta_{12} \approx V_{12} \approx -\frac{m_u}{m_c} \cos \theta_{23} \cot \theta_{12}, \quad (39a)$$

$$\beta_{13} \approx V_{13} \approx \frac{m_u}{m_t} \frac{\cot \theta_{12}}{\sin \theta_{23}}, \quad (39b)$$

$$\beta_{23} \approx V_{23} \approx -\frac{m_c}{m_t} \cot \theta_{23}. \quad (39c)$$

All of the RH angles are small and independent of the effective neutrino masses. Note that, not like the LH quark mixing where  $\tan \theta \approx \sqrt{\frac{m_D}{m_s}}$  in two-generation case [15], the RH mixing angles scale linearly with the ratios of the Dirac neutrino masses.

### 3. numerical results

a. *VO* Inserting the parameters in Eq. (37), we have

$$M_1 \approx 8.0 \times 10^8 \text{GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{GeV}, \quad M_3/r \approx 1.5 \times 10^{17} \text{GeV}. \quad (40)$$

The mixing angles are easy to obtain from Eq. (39),

$$\beta_{12} \approx -4.6 \times 10^{-3}, \quad \beta_{13} \approx 3.2 \times 10^{-5}, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \quad (41)$$

b. *LMA* In this case we have nearly the same RH angles as in VO and we find

$$M_1 \approx 1.5 \times 10^6 \text{GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{GeV}, \quad M_3/r \approx 2.8 \times 10^{14} \text{GeV}. \quad (42)$$

c. *LOW* We now have:

$$M_1 \approx 1.5 \times 10^7 \text{GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{GeV}, \quad M_3/r \approx 6.6 \times 10^{15} \text{GeV} \quad (43)$$

and

$$\beta_{12} \approx -3.3 \times 10^{-3}, \quad \beta_{13} \approx 2.3 \times 10^{-5}, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \quad (44)$$

### B. Case II: small $\theta_{12}$ (SMA)

In this case,  $U_{e3} = 0$  and  $U_{e2}$ ,  $U_{\mu 1}$  and  $U_{\tau 1}$  have the same order  $10^{-2}$  while the other elements of  $U$  are of order 1. We have

$$A \approx U_{e1}^2 \exp(-2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) + U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) \approx X_{11}, \quad (45a)$$

$$B \approx U_{\tau 2}^2 \exp(-2\kappa_3 + 2\kappa_8 + 4\xi_8) + U_{\tau 1}^2 \exp(2\kappa_3 + 2\kappa_8 + 4\xi_8) \approx Y_{33}. \quad (45b)$$

Again, they satisfy  $B > A \gg 3$  and  $A^2 \gg 4B$ . So that

$$Q_1 \approx A \approx f^{-1} U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8), \quad (46a)$$

$$Q_2 \approx \frac{B}{A} \approx U_{\mu 3}^2 \exp(e^{4\kappa_3} + 4\kappa_8 - 2\xi_3 + 2\xi_8), \quad (46b)$$

$$Q_3 \approx \frac{1}{B} \approx f U_{\tau 1}^2 \exp(-2\kappa_3 - 2\kappa_8 - 4\xi_8). \quad (46c)$$

Here  $f = \frac{r}{r + \cot^2 \theta_{12}}$  and it cannot be omitted since  $\cot \theta_{12} \gg 1$ . Similar with case I, we have

$$M_1 \approx f \frac{1}{\sin^2 \theta_{12}} \frac{m_u^2}{m_2^{\text{eff}}}, \quad M_2 \approx \frac{1}{\sin^2 \theta_{23}} \frac{m_c^2}{m_3^{\text{eff}}}, \quad M_3 \approx f^{-1} \sin^2 \theta_{23} \sin^2 \theta_{12} \frac{m_t^2}{m_1^{\text{eff}}}. \quad (47)$$

For the mixing angles, we obtain

$$\beta_{12} \approx V_{12} \approx -f \frac{m_u}{m_c} \cos \theta_{23} \cot \theta_{12}, \quad (48a)$$

$$\beta_{13} \approx V_{13} \approx f \frac{m_u \cot \theta_{12}}{m_t \sin \theta_{23}}, \quad (48b)$$

$$\beta_{23} \approx V_{23} \approx -\frac{m_c}{m_t} \cot \theta_{23}. \quad (48c)$$

Again the factor  $f$  appears. Note that the expressions of  $M_2$  and  $\beta_{23}$  are the same as that when  $\theta_{12}$  is large. Moreover, the SK data suggests strongly that  $\theta_{23} \approx \frac{\pi}{4}$ . So both  $M_2$  and  $\beta_{23}$  have the same values in all the favored regions considered. It is noteworthy that the factor  $f$  makes the value of  $M_3$  remain at a relative low scale for a wide range of  $r$  which is different with that in Ref. [8]. When  $r \gg \cot^2 \theta_{12}$  (then  $f \approx 1$ ), we have the same expressions of the RH masses and the mixing angles no matter whether  $\theta_{12}$  is large or not.

Substituting the values of the parameters in, from Eq. (47) we have

$$M_1 \approx 4.7 \times 10^8 f \text{ GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{ GeV}, \quad M_3 \approx 3.0 \times 10^{12} \frac{r}{f} \text{ GeV}. \quad (49)$$

and from Eq. (48a),

$$\beta_{12} \approx -7.0 \times 10^{-2} f, \quad \beta_{13} \approx 4.9 \times 10^{-4} f, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \quad (50)$$

Here, with the value of  $\theta_{12}$  substituted in,  $f \approx \frac{r}{r + 6.6 \times 10^2}$ .

Comparisons with the exact numerical results are given in Tables II-IV and from which we can see they fit well. In calculation we take  $m_D^{\text{diag}}(\mu)$  at  $\mu = 10^9 \text{ GeV}$ . Note that, although the up quark masses are running with  $\mu$ , the dirac mass hierarchies,  $\eta_3$  and  $\eta_8$ , are almost fixed when  $\mu$  varies. We find they satisfy the following approximate relation

$$\frac{m_u(\mu) m_t(\mu)}{m_c^2(\mu)} \approx 1. \quad (51)$$

So the deviation is mainly resulted from  $F (= \frac{m_t^2}{m_3^{\text{eff}}} e^{4\kappa_8 - 4\xi_8})$  when  $m_D^{\text{diag}}(\mu)$  at different scale is taken.

## V. SUMMARY AND DISCUSSION

In this paper, we introduce a parameterization which transforms all the involving masses in the seesaw formula to the mass ratios. Then by taking the traces of  $X$  and its inverse, we derive the equations of the Majorana mass ratios,  $\eta_3$  and  $\eta_8$ . The solutions to these equations are obtained under some conditions and the elements of  $V_0$  are expressed in a unified form. Assuming quark-lepton symmetry and hierarchical effective neutrino masses, rather simple relations among the various neutrino parameters entering the seesaw formula are deduced. Finally, setting the Dirac neutrino masses to be equal to the up quark masses, we present the numerical results in the favored regions of the solar and atmospheric neutrino experiments.

Now let us give a combined analysis of the results obtained and list our main points as follows:

- $M_2 (\approx 4.6 \times 10^9 \text{ GeV})$  and so the product of  $M_1$  and  $M_3$  are nearly independent of  $\theta_{12}$ .
- The three RH neutrino masses are hierarchical and  $\frac{M_3}{M_2} \left( \propto \frac{m_3^{\text{eff}}}{m_1^{\text{eff}}} \right) \gg \frac{M_2}{M_1} \left( \propto \frac{m_1^{\text{eff}}}{m_2^{\text{eff}}} \right)$ .
- $\beta_{23}$  ( $\approx -4.3 \times 10^{-3}$ ) and  $\beta_{12}/\beta_{13} \approx -\frac{1}{2} \frac{m_t}{m_c} \sin 2\theta_{23} \approx -\frac{1}{2} \frac{m_t}{m_c}$  are also independent of  $\theta_{12}$ . Moreover, the RH mixing angles satisfy the following condition

$$\frac{\beta_{12}\beta_{23}}{\beta_{13}} \approx \cos^2 \theta_{23} \approx \frac{1}{2} \quad (52)$$

which is independent of not only  $\theta_{12}$  and the effective neutrino masses but also the Dirac masses of neutrinos. It is interesting to notice that the (13) elements ( $U_{e3}$ ,  $V_{13}$  and  $U_{us}$ ) determined by the third mixing angles of the three corresponding mixing matrices are all small. It is also noteworthy that the third mixing angles in both the CKM matrix of quarks and the RH mixing matrix are of orders of the products of the other two angles respectively. In the former, we have  $\left| \frac{U_{us}U_{ub}}{U_{cb}} \right| \approx (\rho^2 + \eta^2)^{-\frac{1}{2}}$ . Here,  $\rho$  and  $\eta$  are smaller than one [16].

- Numerically, the lightest right-handed neutrino mass can lie between  $10^6\text{GeV}$  and  $10^8\text{GeV}$  while the heaviest right-handed neutrino mass range from about  $10^{12}\text{GeV}$  to far larger than  $10^{17}\text{GeV}$ .
- Numerically, all the three RH angles are small although they may contain the contribution from the diagonalization of  $M^{-1}$ . The absolute values of  $\beta_{12}$  and  $\beta_{13}$  are about  $10^{-3} \sim 10^{-2}$  and  $10^{-6} \sim 10^{-4}$  respectively.
- SMA solution seems especially attractive in the sense that  $M_3 \sim 10^{15}\text{GeV}$  for a wide range of  $r$  due to the factor  $f$  while  $M_3$ 's for the other three regions (VO, LMA and LOW) increase rapidly with  $r$  and become too large to be viable. Especially, for the VO solution to the solar neutrino problem, both the two mass squared differences splittings (of the order  $10^{-3}\text{eV}^2$  and  $10^{-11}\text{eV}^2$  respectively) and the scale of the heaviest RH neutrino mass  $M_3$  ( $\gg 10^{17}\text{GeV}$ ) make it look very unnatural.

In this work, we have set  $\theta_{13} = 0$ . Although the small  $\theta_{13}$  has little effect on the oscillation solution to the solar and the atmospheric neutrino deficits, it may become important in the seesaw mechanism especially in the SMA region where  $\theta_{13}$  is comparable with  $\theta_{12}$ . It may lead to large RH mixing angles owing to the contribution from the diagonalization of  $M^{-1}$  as well as degenerate masses. This can also be seen from that the coefficient of  $U_{e3}$  in  $A$  are much larger than that of the other elements of  $U$ . We point out that the method is even valid in such case while more skills may be needed. We will discuss it in more details in later paper.

## ACKNOWLEDGMENTS

The authors would like to express our sincere thanks to Professor T. K. Kuo for pointing out this problem to G. Cheng during his visit at Purdue University from January to April, 2000 and provoke our interesting in it. We are also indebted to him for his warmly help in the research progressing and kindness by giving us his papers being prepared. We are

grateful to Dr. Du Taijiao and Dr. Tu Tao for useful discussions. The authors are also grateful to the convenience provided by Chinese High Performance Computing Center at Hefei. The authors are supported in part by the National Science Foundation in China grant no.19875047.



## REFERENCES

- [1] Super-Kamiokande Collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett. **81**, 1158 (1998); *ibid.* **82**, 1810 (1999); Y. Suzuki, <http://lp99.slac.stanford.edu/program.html>.
- [2] Y. Fukuda *et al.*, Phys. Rev. Lett. **81**, 1562 (1998); W.A. Mann, hep-ex/9912007.
- [3] S.M. Bilenky and S.T. Petcov, Rev. Mod. Phys. **59**, 671 (1987).
- [4] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, eds. P. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979) T. Yanagida, in *Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe*, eds. O. Sawada and A. Sugamoto (KEK, 1979) R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980)
- [5] A.Yu. Smirnov, Phys. Rev. **D 48**, 3264 (1993); Nucl. Phys. **B 466**, 25 (1996).
- [6] D. Falcone, Phys. Rev. **D 61**, 097302 (2000); Phys. Lett **B 479**,1 (2000).
- [7] M. Tanimoto, Phys. Lett. **B 345**, 477 (1995); M. Matsuda and M. Tanimoto, Phys. Rev. **D 58**, 093002 (1998); M. Bando, T. Kugo, and K. Yoshioka, Phys. Rev. Lett. **80**, 3004 (1998); G. Altarelli, F. Feruglio, and I. Masina, Phys. Lett. **B 472**, 382 (2000); E. Kh. Akhmedov, G. C. Branco, and M. N. Rebelo, Phys. Lett. **B 478**, 215 (2000).
- [8] T.K. Kuo, Guo-Hong Wu and Sadek W. Mansour, Phys. Rev. **D 61**, 111301 (2000).
- [9] N. Haba and N. Okamura, Eur. Phys. J. **C 14**, 347 (2000).
- [10] T.K. Kuo, Guo-Hong Wu and Shao-Hsuan Chiu, Phys. Rev. **D 62**, 051301 (2000).
- [11] J. Bahcall, P. Krastev, and A.Yu. Smirnov, Phys.Rev. **D 58**, 096016 (1998); *ibid.* **D 60**, 093001 (1999).
- [12] L. Wolfenstein, Phys. Rev. **D 17**, 2369 (1978); S.P. Mikheyev and A.Yu. Smirnov, Nuovo Cimento **C 9**, 17 (1986).
- [13] M. Apollonio *et al.*, Phys. Lett. **B 466**, 415 (1999); F. Boehm *et al.*, hep-ex/0003022; E. Kh. Akhmedov, G. C. Branco, and M. N. Rebelo, Phys. Rev. Lett. **84**, 3535 (2000).
- [14] H. Fusaoka and Y. Koide, Phys. Rev. **D 57**, 3986 (1998).
- [15] C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood Academic Publishers, 1993) p.113; A.Yu. Smirnov, hep-ph/9901208.
- [16] H. Fritzsch and Zhi-zhong Xing, hep-ph/9912358.
- [17] E. Kh. Akhmedov, G.C. Branco, and F.R. Joaquim, Phys. Lett. **B 498**, 237 (2001)

# TABLES

Table I:  $\nu_e \rightarrow \nu_\mu$  solutions to the solar neutrino problem. Here MSW and VO refer to Mikheyev-Smirnov-Wolfenstein matter-enhanced oscillations [12] and vacuum oscillations (so-called just-so oscillation) respectively. LMA (SMA) stands for a large (small) mixing angle and LOW stands for low probability (or low mass).

Solution	$\Delta m_{solar}^2 \text{ (eV}^2\text{)}$	$\sin^2 2\theta_{12}$
VO	$6.5 \times 10^{-11}$	0.75
MSW(LMA)	$1.8 \times 10^{-5}$	0.76
MSW(LOW)	$7.9 \times 10^{-8}$	0.96
MSW(SMA)	$5.4 \times 10^{-6}$	$6.0 \times 10^{-3}$

Table II: Exact numerical and approximate results when  $r = 10^1$ . In each cell we listed the numerical and approximate results above and below respectively. By solving the eigen-equation of  $X$  we obtain the eigenvalue(s) that larger than one and the corresponding eigenvector(s). The reciprocal value(s) of the other eigenvalue(s) and the corresponding eigenvector(s) are obtained by solving the eigen-equation of  $Y$ . Substituting these eigenvalues in Eq.(24) we get the three masses in Majorana sector (see the text for details).

$r = 10^1$	$M_1 \text{ (GeV)}$	$M_2 \text{ (GeV)}$	$M_3 \text{ (GeV)}$	$\beta_{12}$	$\beta_{13}$	$\beta_{23}$
VO	$6.2 \times 10^8$	$4.6 \times 10^9$	$1.9 \times 10^{18}$	$-3.7 \times 10^{-3}$	$2.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$8.0 \times 10^8$	$4.6 \times 10^9$	$1.5 \times 10^{18}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LMA	$1.2 \times 10^6$	$4.5 \times 10^9$	$3.7 \times 10^{15}$	$-3.2 \times 10^{-3}$	$2.2 \times 10^{-5}$	$-4.1 \times 10^{-3}$
	$1.5 \times 10^6$	$4.6 \times 10^9$	$2.8 \times 10^{15}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LOW	$1.3 \times 10^7$	$4.6 \times 10^9$	$7.6 \times 10^{16}$	$-2.6 \times 10^{-3}$	$1.8 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$1.5 \times 10^7$	$4.6 \times 10^9$	$6.6 \times 10^{16}$	$-3.3 \times 10^{-3}$	$2.3 \times 10^{-5}$	$-4.3 \times 10^{-3}$
SMA	$7.0 \times 10^6$	$4.4 \times 10^9$	$2.1 \times 10^{15}$	$-9.3 \times 10^{-4}$	$6.2 \times 10^{-6}$	$-4.0 \times 10^{-3}$
	$7.0 \times 10^6$	$4.6 \times 10^9$	$2.0 \times 10^{15}$	$-1.0 \times 10^{-3}$	$7.2 \times 10^{-6}$	$-4.3 \times 10^{-3}$

Table III: Same as in table I but for  $r = 10^2$ .

$r = 10^2$	$M_1$ (GeV)	$M_2$ (GeV)	$M_3$ (GeV)	$\beta_{12}$	$\beta_{13}$	$\beta_{23}$
VO	$7.7 \times 10^8$	$4.6 \times 10^9$	$1.5 \times 10^{19}$	$-5.3 \times 10^{-3}$	$3.1 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$8.0 \times 10^8$	$4.6 \times 10^9$	$1.5 \times 10^{19}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LMA	$1.5 \times 10^6$	$4.6 \times 10^9$	$2.9 \times 10^{16}$	$-4.4 \times 10^{-3}$	$3.1 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$1.5 \times 10^6$	$4.6 \times 10^9$	$2.8 \times 10^{16}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LOW	$1.4 \times 10^7$	$4.6 \times 10^9$	$6.7 \times 10^{17}$	$-3.2 \times 10^{-3}$	$2.3 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$1.5 \times 10^7$	$4.6 \times 10^9$	$6.6 \times 10^{17}$	$-3.3 \times 10^{-3}$	$2.3 \times 10^{-5}$	$-4.3 \times 10^{-3}$
SMA	$6.1 \times 10^7$	$4.4 \times 10^9$	$2.4 \times 10^{15}$	$-9.1 \times 10^{-3}$	$6.0 \times 10^{-5}$	$-4.0 \times 10^{-3}$
	$6.1 \times 10^7$	$4.6 \times 10^9$	$2.3 \times 10^{15}$	$-9.1 \times 10^{-3}$	$6.4 \times 10^{-5}$	$-4.3 \times 10^{-3}$

Table IV: Same as in table I but for  $r = 10^3$ .

$r = 10^3$	$M_1$ (GeV)	$M_2$ (GeV)	$M_3$ (GeV)	$\beta_{12}$	$\beta_{13}$	$\beta_{23}$
VO	$7.9 \times 10^8$	$4.6 \times 10^9$	$1.5 \times 10^{20}$	$-5.5 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$8.0 \times 10^8$	$4.6 \times 10^9$	$1.5 \times 10^{20}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LMA	$1.5 \times 10^6$	$4.5 \times 10^9$	$2.8 \times 10^{17}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$1.5 \times 10^6$	$4.6 \times 10^9$	$2.8 \times 10^{17}$	$-4.6 \times 10^{-3}$	$3.2 \times 10^{-5}$	$-4.3 \times 10^{-3}$
LOW	$1.5 \times 10^7$	$4.6 \times 10^9$	$6.6 \times 10^{18}$	$-3.3 \times 10^{-3}$	$2.3 \times 10^{-5}$	$-4.3 \times 10^{-3}$
	$1.5 \times 10^7$	$4.6 \times 10^9$	$6.6 \times 10^{18}$	$-3.3 \times 10^{-3}$	$2.3 \times 10^{-5}$	$-4.3 \times 10^{-3}$
SMA	$2.8 \times 10^8$	$4.5 \times 10^9$	$5.1 \times 10^{15}$	$-4.4 \times 10^{-2}$	$2.9 \times 10^{-4}$	$-4.1 \times 10^{-3}$
	$2.8 \times 10^8$	$4.6 \times 10^9$	$5.0 \times 10^{15}$	$-4.2 \times 10^{-2}$	$2.9 \times 10^{-4}$	$-4.3 \times 10^{-3}$